§ topological quantum field theory (TQFT)

Atiyah, Segal around '90

Def. d-dimensional TQFT is the following assignment:

○ Σ: $(d-1)$ dim'l oriented compact mfd without bdry

$\longmapsto Z(\Sigma)$: $\mathbb{C}$-vector space

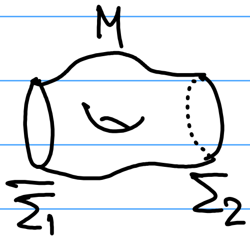
s.t. • $Z(\Sigma \sqcup \Sigma') = Z(\Sigma) \otimes Z(\Sigma')$

• $Z(\phi) = \mathbb{C}$

○ "cobordism"

M : d-dim'l oriented compact mfd

with "in" bdry Σ_1, "out" bdry Σ_2



$\longmapsto$ $Z(M) : Z(\Sigma_1) \longrightarrow Z(\Sigma_2)$ linear map

(M: no bdry $\Rightarrow$ $Z(M) : \underset{\mathbb{C}}{\overset{\parallel}{Z(\phi)}} \longrightarrow \underset{\mathbb{C}}{\overset{\parallel}{Z(\phi)}}$) $\in \mathbb{C}$

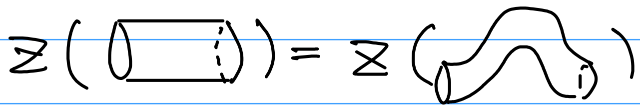
s.t. • equivalent cobordism ($M \xrightarrow{\varphi} M'$ diffeo.

s.t. $\cup$ $\cup$ $\overline{\Sigma}_1, \Sigma_2$ $\varphi|_{\Sigma_i} = id_{\Sigma_i}$)

$\Rightarrow Z(\Sigma_1) \longrightarrow Z(\Sigma_2)$

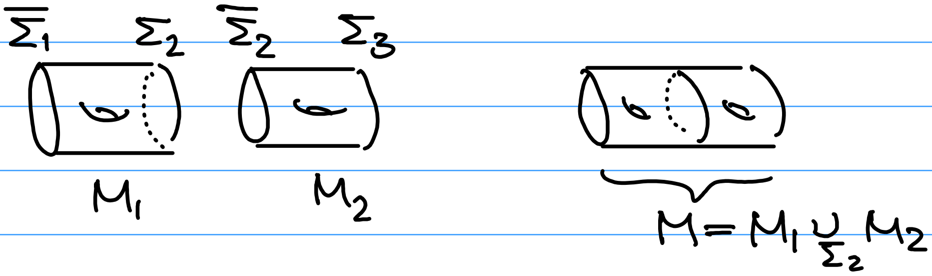
$Z(M) = Z(M')$

i.e.



• $Z(\Sigma \times [0,1]) = id_{Z(\Sigma)}$

•



$$Z(M) : Z(\Sigma_1) \longrightarrow Z(\Sigma_3)$$

$$Z(M_1) \searrow \quad \circlearrowleft \quad \nearrow Z(M_2)$$

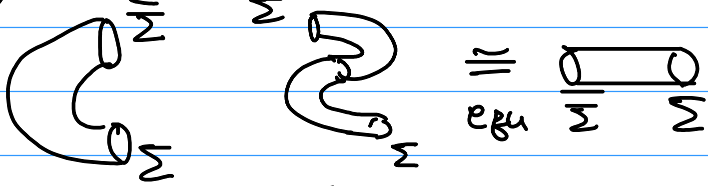
$$Z(\Sigma_2)$$

tensor

- $Z(M \sqcup M') : Z(\Sigma_1) \otimes Z(\Sigma_1') \longrightarrow Z(\Sigma_2) \otimes Z(\Sigma_2')$
  $\quad\; \| \; Z(M) \otimes Z(M')$

Consider  $\quad Z(\overline{\Sigma}) \otimes Z(\Sigma) \xrightarrow{\langle\,,\,\rangle} \mathbb{C}$ pairing

$\gamma : \mathbb{C} \longrightarrow Z(\overline{\Sigma}) \otimes Z(\Sigma)$



$$\therefore \quad id_{Z(\Sigma)} = (\langle\,,\,\rangle \otimes id_{Z(\Sigma)}) \circ (id_{Z(\Sigma)} \otimes \gamma)$$

$$\gamma(1) = \sum u_i \otimes v_i \quad \Rightarrow \quad v = \sum \langle v, u_i \rangle v_i$$

Prop $\quad Z(\Sigma)$ is finite dim'l and $\langle\,,\,\rangle$ is non-degenerate pairing $\quad \therefore Z(\overline{\Sigma}) = Z(\Sigma)^*$

$Z(\Sigma)$ is called the quantum Hilbert space ass. with Σ

This axiomatic approach was introduced by Segal, Atiyah so that mathematicians could understand

Witten's topological quantum field theory

based on usual quantum field theory

§ Intuitive idea behind the axiomatic approach

X : target space (nothing to do with Σ, M etc)
(σ - model)

Suppose $L : \mathrm{Map}(M, X) \longrightarrow \mathbb{R}$ action
lagrangian $\quad \varphi$

e.g. $M = S^1 = \mathbb{R}/\mathbb{Z}$
$L(\varphi) = \int_0^1 \left|\frac{d}{dt} c\right| dt$

$$\rightsquigarrow Z(M) \text{ "}=\text{" } \int_{\mathrm{Map}(M,X)} e^{-L(\varphi)} D\varphi \in \mathbb{C}$$

Feynmann path integral

Next suppose M has "out" bdry Σ $\qquad f : \Sigma \longrightarrow X$

$$Z(M, f) \text{ "}=\text{" } \int_{\substack{\mathrm{Map}(M,X) \ni \varphi \\ \varphi|_\Sigma = f}} e^{-L(\varphi)} D\varphi$$

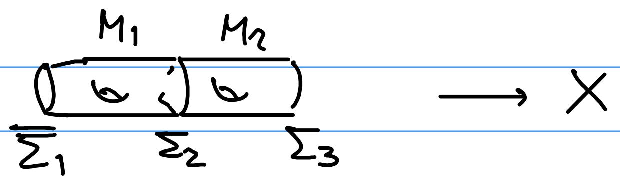
We regard this as a function

$$\begin{array}{ccc} f & \longmapsto & Z(M, f) \\ \cap & & \cap \\ \mathrm{Map}(\Sigma, X) & \longrightarrow & \mathbb{C} \end{array}$$

i.e. $Z(M, \bullet) \in \mathrm{Function}(\mathrm{Map}(\Sigma, X))$

$\therefore$ $\mathbb{C}$ - vector space

Recall $Z(\Sigma)$ quantum Hilbert space cobordism



$f_1 : \Sigma_1 \to X \qquad\qquad f_3 : \Sigma_3 \to X$

$$Z(M_1 \cup_{\Sigma_2} M_2, f_1, f_3) = \int_{\substack{\mathrm{Map}(M,X) \ni \varphi \\ \varphi|_{\Sigma_1} = f_1,\ \varphi|_{\Sigma_3} = f_3}} e^{-L(\varphi)} D\varphi$$

Note :

$$\mathrm{Map}(M_1 \cup_{\Sigma_2} M_2, X) = \mathrm{Map}(M_1, X) \underset{\mathrm{Map}(\Sigma_2, X)}{\times} \mathrm{Map}(M_2, X)$$

$$= \{ (\varphi_{12}, \varphi_{23}) \in \mathrm{Map}(M_1, X) \times \mathrm{Map}(M_2, X) \mid \varphi_{12}|_{\Sigma_2} = \varphi_{23}|_{\Sigma_2} \}$$

$\therefore\ Z(M_1 \cup_{\Sigma_2} M_2, f_1, f_3)$

$$= \int Df_2 \underbrace{\int_{\substack{\mathrm{Map}(M_1, X) \ni \varphi_{12} \\ \varphi_{12}|_{\Sigma_1} = f_1 \quad \varphi_{12}|_{\Sigma_2} = f_2}} e^{-L(\varphi_{12})} D\varphi_{12}}_{Z(M_1, f_1, f_2)} \underbrace{\int_{\substack{\mathrm{Map}(M_2, X) \ni \varphi_{23} \\ \varphi_{23}|_{\Sigma_3} = f_3 \\ \varphi_{23}|_{\Sigma_2} = f_2}} e^{-L(\varphi_{23})} D\varphi_{23}}_{Z(M_2, f_2, f_3)}$$

$$= \int_{\mathrm{Map}(\Sigma_2, X)} Df_2\, Z(M_1, f_1, f_2) \cdot Z(M_2, f_2, f_3)$$

(cf. convolution product of functions

1-dim'l QFT = Quantum mechanics

$M^1 \longrightarrow X$ …. motion of particles

Σ^0 : 0-dim'l mfd e.g. $\{0\}$ $\quad$ $Map(\{0\}, X) \cong X$

$Func(Map(\{0\}, X)) = L^2(X)$

$\longrightarrow$ This case can be path-integral can be rigorously justified

But $Z(\Sigma)$ is a finite dim'l vector space $\neq L^2(X)$

Q. Why is topological QFT meaningful?

A. TQFT is a toy model for supersymmetric QFT's

Replace $L^2(X) \rightsquigarrow \Omega^*_{L^2}(X)$: "L^2"-differential form

$\circlearrowleft d$: exterior differential s.t. $d^2 = 0$

$$Z(M) : Z(\overline{\Sigma_1}) \longrightarrow Z(\Sigma_2)$$

$$\overset{\text{1 mfd}}{\ni} \qquad (\Omega^*_{L^2}(X))^{\otimes \# \overline{\Sigma_1}} \qquad (\Omega^*_{L^2}(X))^{\otimes \# \Sigma_2}$$

$\supset \circ \subset$

supersymmetry : $Z(M)$ commutes with d

$$d\, Z(M) = Z(M)\, d$$

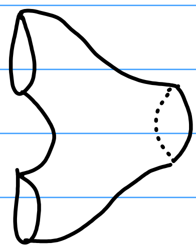
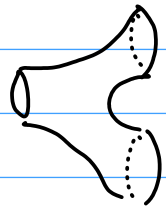
Then $Z(M)$ induces $\mathrm{Ker}\, d / \mathrm{Im}\, d \longrightarrow \mathrm{Ker}\, d / \mathrm{Im}\, d$

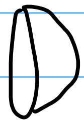
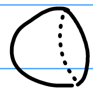
$H^*(X)^{\otimes \cdots}$ $\quad$ $H^*(X)^{\otimes \cdots}$

finite dim'l

§ $d=2$ TQFT $\quad Z(S^1) =: V$: vector space

note $\quad \overline{S^1} = S^1$

 : $V \otimes V \xrightarrow{\mu} V$ $\qquad$  : $V \xrightarrow{\Delta} V \otimes V$

 ; $V \xrightarrow{\varepsilon} \mathbb{C}$ $\qquad$  : $\mathbb{C} \xrightarrow{1} V$

<u>Folklore theorem</u> $d=2$ TQFT $\longleftrightarrow$ commutative Frobenius algebra

cf. Kock

<u>Def.</u> $(V, 1, \mu, \varepsilon)$: comm. Frobenius algebra

- V : $\mathbb{C}$-vectr sp finite dim'l
- μ : commutative multiplication on V
  s.t. 1 is unit
- $\varepsilon : V \longrightarrow \mathbb{C}$ Frobenius form
  s.t. $(\,,\,) = \varepsilon \circ \mu : V \otimes V \longrightarrow \mathbb{C}$
  is non-degenerate

Δ is the dual of μ via $(\,,\,)$

§

I am recently studying Coulomb branches of gauge theories.

This is partially motivated by $d=3$ TQFT

$A = Z(S^2)$ is a commutative algebra

but I do not assume A is finite dimensional

$$\left(Z(S^2 \sqcup S^2) = Z(S^2) \hat{\otimes} Z(S^2) \right)$$

completed tensor product

Coulomb branch $\mathcal{M}$ is an affine algebraic variety ($/\mathbb{C}$)

defined as $\mathcal{M} = \operatorname{Spec} A$

or $\mathbb{C}[\mathcal{M}] = A$
ring of algebraic functions

Originally Coulomb branch was studied by physicists without mathematically rigorous definition.